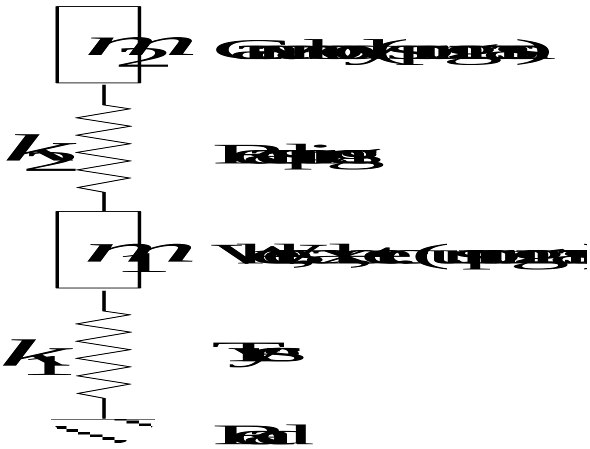
MULTI-DEGREE-OF-FREEDOM SYSTEMS

MODULE: MMME2046 DYNAMICS & CONTROL

Many structures can be approximated by several rigid bodies connected by flexible elements. For example, the body of a caravan is connected to the axle by suspension springs and the axle is connected to the road by the flexible tyres. A suitable dynamic model is shown below.



The body and the axle can move separately from each other, so this model has 2 degrees-of-freedom (2 independent *possible motions*). We will need 2 coordinates to describe how the system moves; axle displacement and body displacement.

We will see that the model will predict 2 natural frequencies, each of which will have a characteristic pattern of displacement, called a ***mode shape***.

DEFINITION:

Mode shape Characteristic motion (deflection) pattern for a structure vibrating at one of its natural frequencies

The demonstration system exhibits similar behaviour. This 2 degree-of-freedom system also has 2 natural frequencies and 2 mode shapes.

We will study 2 classes of structures that have more than one mode of vibration.

Multi-degree-of-freedom systems (which have discrete masses and springs)

Shafts and beams (which have distributed mass and stiffness)

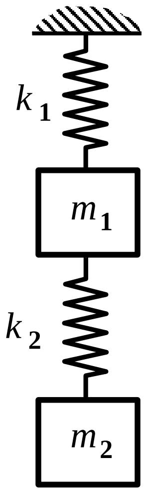
In both cases, the aim will be to calculate

1. The **natural frequencies** of the system
2. The **corresponding mode shapes**

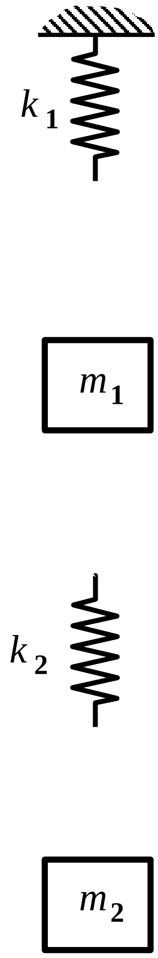
EXAMPLE 1: DEMONSTRATION SYSTEM (2 DEGREES OF FREEDOM)

The approach to these problems is an extension of the one used for single-degree-of-freedom systems, except that we will have a separate free body diagram for each of the rigid elements. Each will also have its own equation of motion.

STEP 1: Dynamic Mass-Spring Model



STEP 2: Free-body Diagrams



The free-body diagram shows a "snapshot" of the system when x_1 and x_2 are **both positive**. The arrows on the diagram need to show the **positive direction** of the forces that each spring exerts on the masses. The expression for each force is given by

Spring force = Stiffness x Change of length.

Take each spring in turn and ask yourself the following questions.

- What is the **change of length** of the spring?
- Is the spring in **tension** or **compression**?

For the change of length, look at the positive displacements of the two ends of the spring. If the direction of movement at both ends is the **same** (as it is in this example), the change of length will be the **difference** between the individual displacements. If, however, the chosen positive directions for the two ends of the spring are **different**, the change of length will be the **sum** of the individual displacements. You need to be systematic when setting up the free body diagrams.

STEP 3: Equations of motion

or

(1a)

(1b)

In matrix form,

or $$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

As with single-degree-of-freedom systems, we can check for errors in the equations at this stage. In particular,

(1) the terms in the leading diagonals of $[M]$ and $[K]$ are **always positive**
(2) the off-diagonal terms may be positive or negative
(3) $[M]$ and $[K]$ are **often symmetric** about the leading diagonal

The equations are **coupled**; each involves both x_1 and x_2. Physically, the coupling tells us that if one mass moves, the other mass will also move; push the top mass down and the lower mass moves as well. Put another way, motion of one mass cannot occur independently of the other. Mathematically, the coupling means that the equations must be solved simultaneously.

For free vibration of the system at one of its natural frequencies, the motion of each mass will be sinusoidal. Use as substitutions, $x_1(t) = X_1 \cos\omega t$ and $x_2(t) = X_2 \cos\omega t$.

Substituting into (1) gives

or in matrix form

(2a)

(2b)

or $$\left([K] - \omega^2 [M]\right)\{X\} = \{0\} \qquad (3)$$

or just $$[Z]\{X\} = \{0\}$$

where $$[Z] = [K] - \omega^2 [M]$$

Note that $[Z]$ is often called the ***Dynamic Stiffness Matrix***.

You should recognise equation (3) as an eigenvalue problem; normally presented in maths text books in the form

$$\left([A] - \lambda [B]\right)\{X\} = \{0\}$$

The eigenvalues give the natural frequencies and the eigenvectors give the corresponding mode shapes.

You will have met various ways of solving eigenvalue problems, most of which are appropriate to computer-based solution because of the weight of algebra involved. In this module, problems will have 2 or 3 degrees-of-freedom. These lead to 2x2 or 3x3 matrices. In this case, the simplest way to the solutions is as follows (use other methods if you prefer).

For a non-trivial solution of equation (3), $\det[Z] = 0$

Multiplying out the determinant gives

(4)

Equation (4) is called the ***Frequency Equation***. For this problem, it is a quadratic in ω^2 and will have two roots, ω_{n1}^2 and ω_{n2}^2, where ω_{n1} and ω_{n2} are the two natural frequencies of the system and gives us the first part of the information we are looking for.

To find the corresponding mode shapes, we substitute each root back into equation (2a) or (2b) to get the relationship between X_1 and X_2. Since (2) is a pair of homogeneous equations, we cannot find unique solutions for X_1 and X_2 separately.

One way of resolving this is to give one value an amplitude of unity and then find the amplitude of the other relative to this. For example, let $X_2 = 1$, using (2b), we get

(5)

The vector $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ is the required *Mode Shape*.

NUMERICAL EXAMPLE: $m_1 = m_2 = 2$ kg $k_1 = k_2 = 200$ N/m

Natural frequencies:
Substituting into (4) and solving gives $\omega_{n1}^2 = 38.1\ s^{-2}$ and $\omega_{n2}^2 = 261.8\ s^{-2}$

Hence, $\omega_{n1} = 6.18$ rad/s = 0.98 Hz and $\omega_{n2} = 16.18$ rad/s = 2.58 Hz

Mode shapes:

Mode 1: Put $\omega_{n1}^2 = 38.1\ s^{-2}$ into (5) to give $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1.0 \end{Bmatrix}$

Mode 2: Put $\omega_{n2}^2 = 261.8\ s^{-2}$ into (5) to give $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1.0 \end{Bmatrix}$

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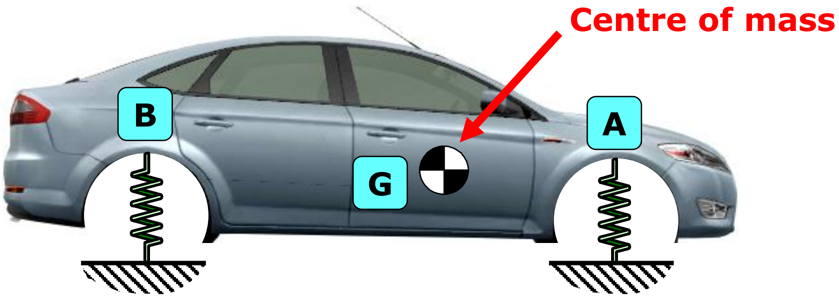
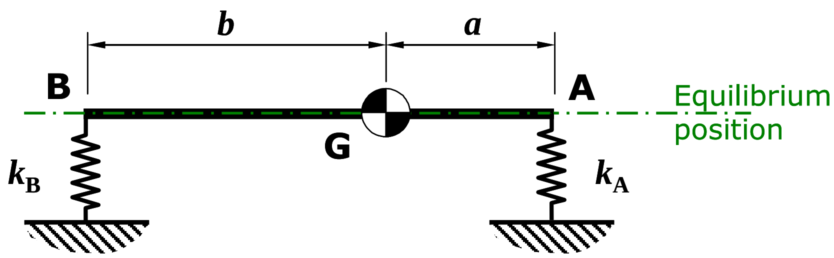
EXAMPLE 2 2D vehicle model
(coupled bounce and pitch)

STEP 1: In developing the dynamic model, a number of assumptions will be made

1. There is no roll motion - pitch and vertical translation only
2. The body is rigid, with mass, m, and moment of inertia, I_G
3. The tyres are very stiff so that the axles do not move
4. k_A and k_B are the combined stiffnesses for the front and rear springs respectively
5. The shock absorbers are ignored





Selection of coordinates

This was easy with the two masses in the first example. Here, we could use x_G (displacement of G) together with θ (pitch angle), or we could use the two displacements x_A and x_B. Either pair will describe the motion of the body.

Q *Does it matter which pair we use ?*
A **NO** - provided the equations of motion are right

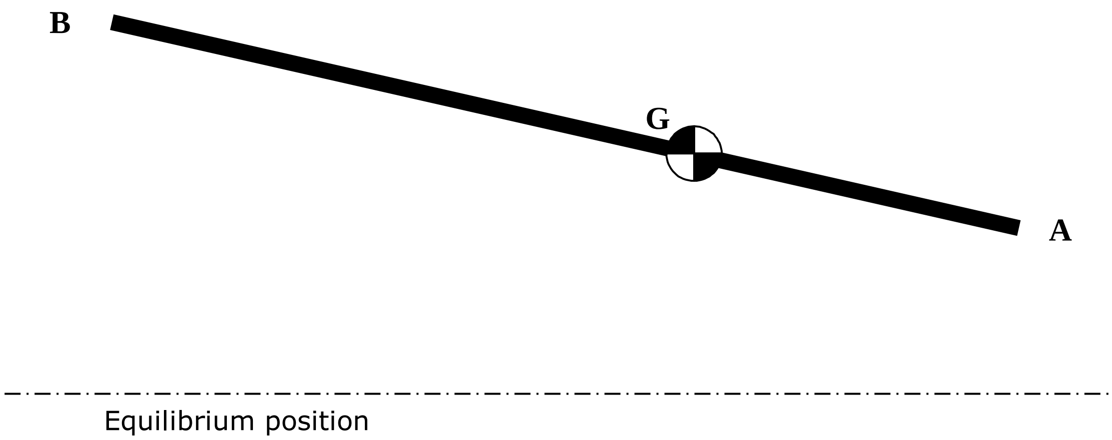
Q *What equations of motion will we have ?*
A **1. vertical translation** – we will need the absolute acceleration of the centre of mass

2. angular motion about G (there is no fixed axis on AB) – we will need the angular acceleration of AB

This suggests x_G and θ would be good choices. If we chose the alternatives of x_A and x_B, we would need to find expressions for x_G and θ in terms of x_A and x_B and then differentiate twice to get the two acceleration terms. This can be done, but is a bit more complicated.

STEP 2: Free Body Diagram

To draw the free body diagram, it is helpful to draw the displaced position of AB. The deflections at A and B can then be seen clearly, making it easier to work out the forces in the springs.



For small θ, the increases in spring lengths are

$$x_A = x_G - a\theta$$

(1)

and $$x_B = x_G + b\theta$$

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NUMERICAL EXAMPLE:

m = 900 kg I_G = 1000 kgm2

k_A = 25 kN/m k_B = 10 kN/m

a = 1 m b = 2 m

Natural frequencies:
The frequency equation becomes

$$\begin{vmatrix} 35\times10^3 - 900\omega^2 & -5\times10^3 \\ -5\times10^3 & 25\times10^3 + 40\times10^3 - 1000\omega^2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 35 - 0.9\omega^2 & -5 \\ -5 & 65 - \omega^2 \end{vmatrix} = 0$$

Expanding,

$$0.9\omega^4 - 93.5\omega^2 + 2250 = 0$$

Roots are $\omega_{n1}^2 = 37.8\ s^{-2}$ and $\omega_{n2}^2 = 66.0\ s^{-2}$

Hence, $\omega_{n1} = 0.98$ Hz and $\omega_{n2} = 1.29$ Hz

Mode shapes:

Mode #1: Put $\omega_{n1}^2 = 37.8\ s^{-2}$ into (3) to give $\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 5.43 \\ 1.0 \end{Bmatrix} \begin{matrix} \text{m} \\ \text{rad} \end{matrix}$

Mode #2: Put $\omega_{n2}^2 = 66.0\ s^{-2}$ into (3) to give $\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} -0.205 \\ 1.0 \end{Bmatrix} \begin{matrix} \text{m} \\ \text{rad} \end{matrix}$

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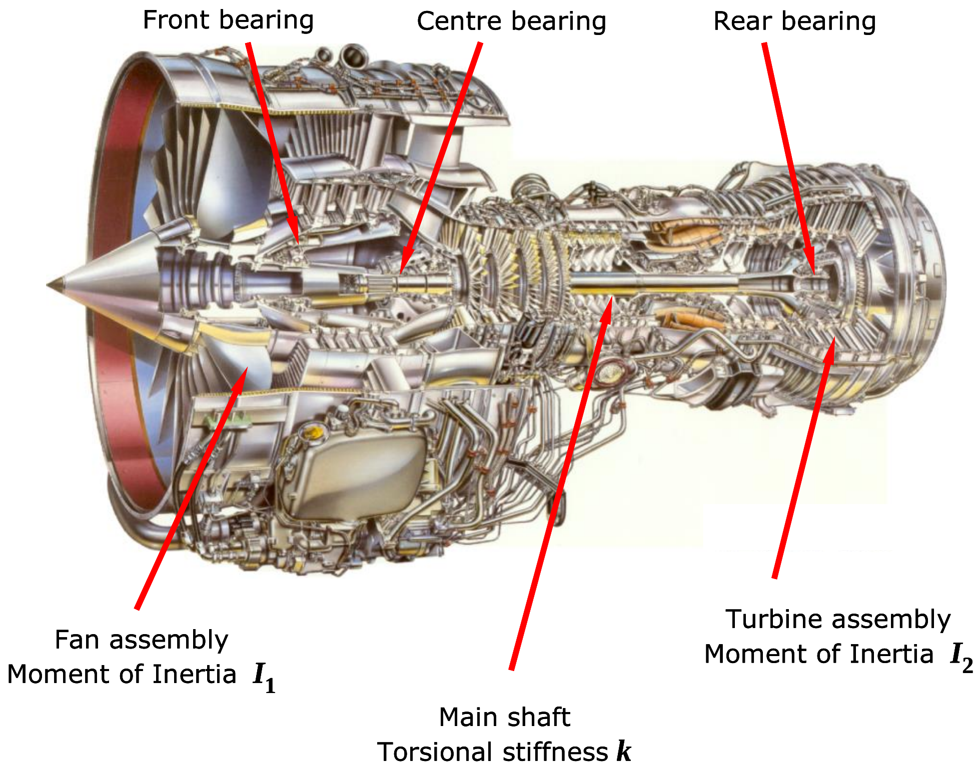
Torsional Systems

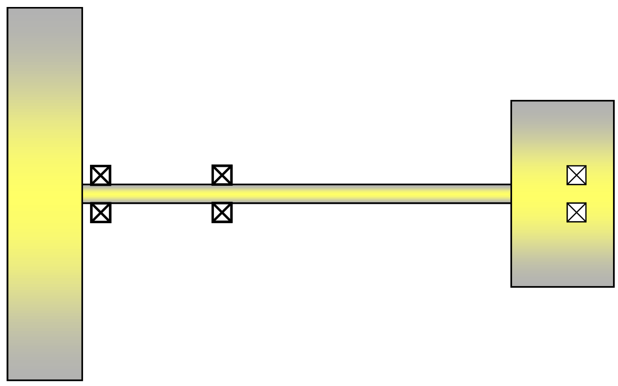
Most mechanical power is transmitted via rotating shafts. Examples include the turbine-alternator sets in power stations, the tail rotor drive in a helicopter and the propeller shaft in a ship's propulsion system. One of the problems for the vibration engineer is that power surges or sudden changes in load can induce transient torsional oscillations that are superimposed on the uniform rotation. Also, the drive torque can sometimes contain fluctuating components that can set up steady state torsional oscillations in the system. In each case, the torsional natural frequencies and modes shapes are required to study the behaviour.

Example Main drive shaft of a gas turbine engine

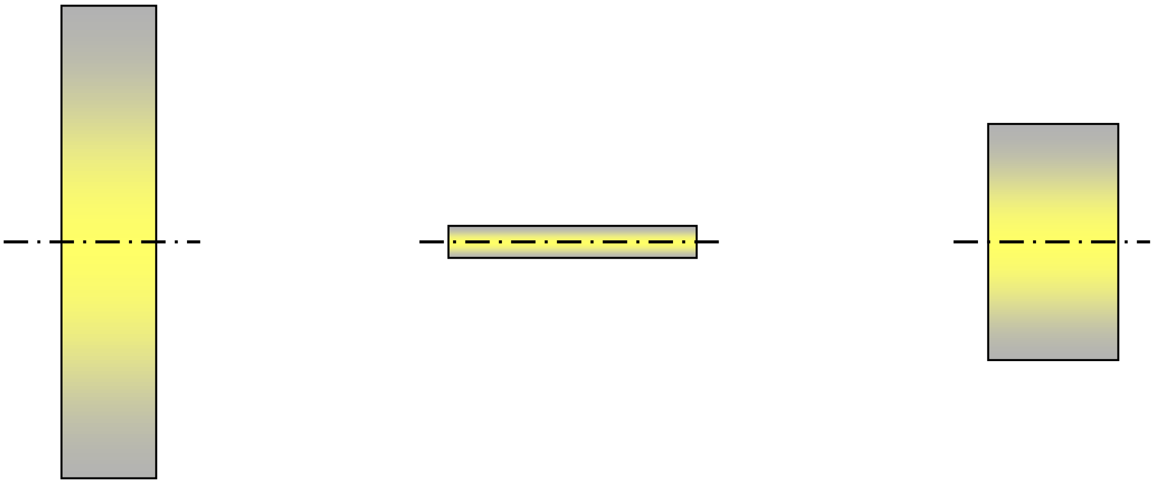
There are two drive shafts in the V2500 engine, which powers the majority of Airbus 320 aircraft (and several others) currently in service. The main shaft transmits power from a 5-stage turbine at the rear to the fan and the first 3 compressor stages. The whole assembly rotates freely, supported by three bearings. The dynamic model will assume that the fan and turbine assemblies can be treated as rigid elements and that the mass of the shaft can be neglected.



Step 1 Dynamic mass-spring model



Step 2 Free body diagrams



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Note on Rigid Body Modes

A Rigid Body mode is characterised by a natural frequency equal to zero and by a mode shape in which there is no deformation of any of the parts; that is, the system moves as if it was a single rigid body.

Any structure that is capable of moving without deformation (this is true of any structure not connected to ground – Sheet 2, Q1(a) and Q1(c) are other examples) WILL have one (or more) rigid body modes with $\omega_n = 0$. It follows that the frequency equation will not contain a constant term. Since you can tell in advance that this should be the case, it's a useful check that the frequency equation is correct.

SUMMARY OF APPROACH

Objective: To find the natural frequencies and the corresponding mode shapes

1. Develop a dynamic model of the structure

2. Draw free body diagrams following the three stages:
   (i) Start with the system in equilibrium and separate it into its constituent free bodies by drawing them without any of the restraining springs. The forces exerted by the springs are added in stage (iii).
   (ii) Select motion coordinates to describe how the system will deflect from its equilibrium position and mark them on the diagram showing your chosen positive directions.
   (iii) Apply positive deflections to all of the chosen motion coordinates, identify the forces (and/or moments) that result and draw them on the diagrams.

3. Write down the equations of motion
   These will be in the form $\left[M \right] \left\{ \ddot{x} \right\} + \left[K \right] \left\{ x \right\} = \left\{ 0 \right\}$

4. **Check the form of the equations**
   The expressions for the leading diagonal terms in $\left[M \right]$ and $\left[K \right]$ **MUST be positive**. You **HAVE DEFINITELY** made a mistake if they are not.

   $\left[M \right]$ and $\left[K \right]$ **should NORMALLY be symmetric**. You **MAY** have made a mistake if they are not.

5. Substitute $\left\{ x\left(t \right) \right\} = \left\{ X \right\} \cos\omega t$

6. Write the equations in the form $\left[Z \right]\left\{ X \right\} = \left\{ 0 \right\}$
   You can skip step 5 and write $\left[Z \right] = \left[K \right] - \omega^2 \left[M \right]$ directly if you wish.

7. Find the frequency equation $\det\left[\mathrm{Z} \right] = 0$
   If the system is not attached to ground anywhere, it will have a rigid body mode, which has a frequency equal to zero. In this case, there will be no constant term in the frequency equation, which is a useful check that the frequency equation is correct.

8. The roots of the polynomial give the natural frequencies.

9. Substitute each root in turn back into $\left[Z \right]\left\{ X \right\} = \left\{ 0 \right\}$ and solve for the mode shape vectors, $\left\{ X \right\}$